

في حل المعادلات التفاضلية باستخدام تحويلات لابلاس

On Solutions of Differential Equations
by using Laplace Transformation

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Abstract:

Our aim in this paper is to find the solution of Linear Simultaneous Ordinary Differential Equations (L.S.O.D.E) with constant coefficients subjected to some initial conditions or without using any conditions by using Laplace Transformation (L.T) through generalized the two methods that Mohammed [1],[2] approached to it.

Introduction:

Laplace Transformation [5] is considered as one of important transformation which is known to solve the Linear Ordinary Differential Equations(L.O.D.E) with constants coefficients .

Mohammed [1] supposed a (L.O.D.E.) of order (n) with constant coefficients and due to certain initial conditions which general form can be written as follows :

$$a_0 y^n + a_1 y^{n-1} + \dots + a_n y = f(x) \quad , \quad (1)$$

where a_0, a_1, \dots, a_n are constants, y^n the n th derivative of the function $y(x)$, $f(x)$ is a continuous function whose L.T can be determined, and $y(0), \dots, y^{n-1}(0)$ are defined. To find a solution of D.E (1) he took L.T to both sides, after substituting initial conditions and simplification he put $L(y)$ as follows

$$L(y) = \frac{h(p)}{k(p)}, \quad (2)$$

where h, k are polynomials of p , such that the degree of h is less than the degree of k and the polynomial k with known prime cofactors, by taking the inverse of Laplace Transformation ($L^{-1}.T$) to both sides of equation(2) he found

$$L(y) = L^{-1}\left\{\frac{h(p)}{k(p)}\right\}, \quad (3)$$

equation(3) represents the general solution of the equation(1) which can be written of the form :

$$y = A_0k_0(x) + A_1k_1(x) + \dots + A_mk_m(x) \quad (4)$$

such that k_0, k_1, \dots, k_m are functions of x and that A_0, A_1, \dots, A_m are constants, whose number equals to the degree of $k(p)$ to find the values of A_0, \dots, A_m he substituted the initial conditions, one of them is $y(0)$ so he found

$$A_0k_0(0) + A_1k_1(0) + \dots + A_mk_m(0) = y(0) \quad (m_0)$$

By taking derivatives of (4) m times he got

$$A_0k_0'(0) + A_1k_1'(0) + \dots + A_mk_m'(0) = y'(0) \quad \dots(m_1)$$

$$A_0k_0''(0) + A_1k_1''(0) + \dots + A_mk_m''(0) = y''(0) \quad \dots(m_2)$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$A_0k_0^{(m)}(0) + A_1k_1^{(m)}(0) + \dots + A_mk_m^{(m)}(0) = y^{(m)}(0) \quad (m_0)$$

This linear system can be solved to obtain A_0, \dots, A_m and so by this method he obtained the solution of the required differential equation(1). Mohammed [2] also, supposed the L.O.D.E. (1) of order (n) with constant coefficients and without subjected to any initial conditions i.e $y(0), y'(0), \dots, y^{(n-1)}(0)$ are unknown and the

L.T of $f(x)$ is known, so to solve equation(1), he took L.T to both sides, he found :

$$L(y) = \frac{k(p)}{(a_0 p^n + a_1 p^{n-1} + \dots + a_n)h(p)},$$

where $h(p)$ is polynomial of (p) represents denominator of L.T of the function $f(x)$ and $k(p)$ is also a polynomial of p and its degree smaller than the degree of the product of $(a_0 p^n + a_1 p^{n-1} + \dots + a_n)$ and $H(p)$ and after he took $(L^{-1}.T)$ to both sides of the above equation, he found that

$$y = A_1 g_1(x) + A_2 g_2(x) + \dots + A_n g_n(x) + B_1 h_1(x) + B_2 h_2(x) + \dots + B_r h_r(x) \dots (5)$$

where A_1, \dots, A_n and B_1, \dots, B_r are constants and g_1, \dots, g_n and h_1, \dots, h_r are functions of x . The number of the constants B_i and the number of the functions $h_i, i = 1, 2, \dots, r$ are equal to the degree of $h(p)$ which is supposed to be (r) .

Note that the order of equation (1) is (n) therefore its general solution must contain (n) constants, but the solution in equation (5) contains $(n + r)$ constants and to solve this problem, he eliminated some of these constants B_1, \dots, B_r by obtaining whose values by substituting the solution (5) in the equation(1) so he found a solution contains only (n) constants (unknown) and by this method he got the general solution of equation(1) without using any initial conditions by using L.T.

In (2007) Noor [3] generalized the above two methods for solving Linear Partial Differential Equations with constants coefficients .

In this paper ,we generalized the two methods which Mohammed [1],[2] approached to it for solving L.S.O.D.E. with constant coefficients either subjected or not subjected to initial conditions.

Note (1):

The number of arbitrary independent constants which can appear in the general solution for example the set of L.S.O.D.E.

$$\begin{aligned} f_1(D)x + g_1(D)y + k_1(D)z &= h_1(t) \\ f_2(D)x + g_2(D)y + k_2(D)z &= h_2(t) \\ f_3(D)x + g_3(D)y + k_3(D)z &= h_3(t), \end{aligned}$$

where D is an operator which is represents $D = \frac{d}{dt}$.

can be equal to the degree of D in the determinate

$$\begin{vmatrix} f_1(D) & g_1(D) & k_1(D) \\ f_2(D) & g_2(D) & k_2(D) \\ f_3(D) & g_3(D) & k_3(D) \end{vmatrix} \equiv \Delta$$

by assuming that Δ is not equal to zero but if $\Delta \equiv 0$ then the set is not independent and we are going to discuss these sets here[5].

Basic Definitions and Properties :

Definition 1:

The Laplace Transformation of the function $f(x)$ such that $x > 0$ and denoted by $L\{f(x)\}$ defined as follows:

$$L\{f(x)\} = \int_0^{\infty} e^{-px} f(x) dx = F(p)$$

provided that this integration is convergent, p is a real number .

Definition 2:

If $F(p)$ represent a Laplace Transformation for a function $f(x)$ then $f(x)$ is said to be inverse of Laplace Transformation and will be written as :

$$f(x) = L^{-1}\{F(p)\}$$

If the Laplace Transformation $f(x)$ is known then directly we can find the inverse of L.T.or by using the method of partial fraction[6].

Property 1:

If $f_1(x), f_2(x), \dots, f_n(x)$ are (n) functions such that $x > 0$ and c_1, \dots, c_n are (n) number then $L\{c_1f_1(x) + c_2f_2(x) + \dots + c_nf_n(x)\} = c_1L\{f_1(x)\} + c_2L\{f_2(x)\} + \dots + c_nL\{f_n(x)\}$ see[4] .

Property 2:

Let $f(x)$ be a function such that $x > 0$, then $L\{e^{bx}f(x)\} = F(p-b)$ Where b is a number . See [4] .

Property 3:

If $L^{-1}\{F_1(p)\} = f_1(x)$, $L^{-1}\{F_2(p)\} = f_2(x)$, ..., $L^{-1}\{F_n(p)\} = f_n(x)$ and if a_1, \dots, a_n are numbers then $L^{-1}\{a_1F_1(p) + a_2F_2(p) + \dots + a_nF_n(p)\} = a_1f_1(x) + a_2f_2(x) + \dots + a_nf_n(x)$ see[4] .

Property 4:

If $L^{-1}\{F(p)\} = f(x)$,then $L^{-1}\{F(p-a)\} = e^{ax} L^{-1}\{F(p)\}$ see[4].

Generalized the Two suggested Methods [1],[2] for Solving (L.S.O.D.E):

Generally, to solve the set of (L.S.O.D.E) let us consider the system of (L.O.D.E)

$$\begin{aligned} f_1(D) x + g_1(D) y + k_1(D) z &= h_1(t) \\ f_2(D) x + g_2(D) y + k_2(D) z &= h_2(t) \\ f_3(D) x + g_3(D) y + k_3(D) z &= h_3(t) , \end{aligned}$$

by taking L.T. to both sides .We can put

$$\begin{aligned} f_1(p) x + g_1(p) y + k_1(p) z &= h_1^*(p) \\ f_2(p) x + g_2(p) y + k_2(p) z &= h_2^*(p) \\ f_3(p) x + g_3(p) y + k_3(p) z &= h_3^*(p) , \end{aligned}$$

where f_i , g_i and k_i are polynomials of p where $i = 1, 2, 3$ and

$h_i^*(i=1,2,3)$ are the L.T of $h_i(i=1,2,3)$ respectively.

Also, we can use Grammar Rule [6] to find $X(p)$:

$$X(p) = \frac{N(p)}{M(p)}, \quad \text{see [2]}$$

where $M(p)$ is a polynomial of p represents denominator of (L.T) of the function $X(t)$ and $N(p)$ is also a polynomial of p and its degree smaller than the degree of the polynomial $M(p)$. Now let us consider that:

I The above system are not subjected to any initial conditions, then by taking $(L^{-1}.T)$ of the above equation, we get

$$x(t) = L^{-1} \left\{ \frac{N(p)}{M(p)} \right\}, \quad \text{see [2]}$$

By the same way, we can find the formulas of $y(t)$ and $z(t)$. So $x(t), y(t)$ and $z(t)$ represent the general solution formula of (L.S.O.D.E) which have contain a number of constants which are more than the degree of Δ , therefore we derive the solution formula which we have got for every variable according to the order of the (L.S.O.D.E) and then we substitute it in the original equations and equalize the coefficients, by doing this we get the values of the extra constants in the solution and we obtain the general solution formula for (L.S.O.D.E).

II- The above system are subjected to some initial conditions then we substitute these conditions before got the formula of $X(p)$. And by taking L^{-1} of (L.T) of equation (1), we get

$$x(t) = L^{-1} \left\{ \frac{N(p)}{M(p)} \right\}, \quad \text{see[1]}$$

And by the same way we can find the formula of $y(t)$ and $z(t)$. So $x(t), y(t)$ and $z(t)$ represent the general solution formula of (L.S.O.D.E) which it have a number of constants and to find the values of these constants we will see that the number of initial conditions are not enough to solve the system of

equations which we have got, because the number of equations may be less than the number of constants, whence we should find additional conditions that can be get by the (L.S.O.D.E) and the initial conditions that are given, for this we may derive the given (L.S.O.D.E) many times and then we get the additional conditions, the requisite importance that the number of equations must equalize the number of constants that appear in the general solution in order to solve the linear system.

Examples:

Example (I) : To solve the set of the L.S.O.D.E

$$X'(t) + Y'(t) + Y(t) = 1$$

$$X'(t) + 2X(t) - Z'(t) + Z(t) = 1$$

$$Y'(t) + Y(t) + Z'(t) + 2Z(t) = 0,$$

by using (L.T) without using any initial conditions, we take (L.T) to both sides of the above system, so we get

$$p X(p) + (p+1)Y(p) = \frac{h_1(p)}{p}$$

$$(p+2) X(p) - (p-1) Z(p) = \frac{h_2(p)}{p}$$

$$(p+1) Y(p) + (p+2) Z(p) = h_3(p) \quad , \text{ see [2]}$$

where $h_1(p), h_2(p)$ and $h_3(p)$ are polynomial of p , by using grammar rule, we get

$$X(p) = \frac{K_1(p)}{p(p+1)(p+\frac{4}{5})}$$

where $K_1(p)$ is a polynomial of p which has degree less than three i.e less than the degree of denominator. Now, we can get the solution $x(t)$ after taking $(L^{-1}.T)$ to both sides, so we get

$$X(t) = A_1 + B_1 e^{-t} + C_1 e^{\frac{-4}{5}t}$$

by the same way we can find $y(t)$ and $z(t)$, which formulas are given by

$$Y(t) = A_2 + B_2 e^{-t} + C_2 e^{\frac{-4}{5}t}$$

$$Z(t) = A_3 + B_3 e^{-t} + C_3 e^{\frac{-4}{5}t}$$

where $x(t)$, $y(t)$ and $z(t)$ represent the general solution of required equations.

The given equations have second degree of Δ , so the general solution must contain two constants only, while the general solution formula of (L.S.O.D.E) has nine constants, therefore, we should eliminate six constants, for this we get $x'(t)$, $y'(t)$ and $z'(t)$ from the general solution, as follows:

$$X'(t) = -B_1 e^{-t} - \frac{4}{5} C_1 e^{\frac{-4}{5}t}$$

$$Y'(t) = -B_2 e^{-t} - \frac{4}{5} C_2 e^{\frac{-4}{5}t}$$

$$Z'(t) = -B_3 e^{-t} - \frac{4}{5} C_3 e^{\frac{-4}{5}t}$$

we substitute $X(t)$, $X'(t)$, $Y(t)$, $Y'(t)$, $Z(t)$ and $Z'(t)$ in the original equations, we get:

$$-B_1 e^{-t} - \frac{4}{5} C_1 e^{\frac{-4}{5}t} - B_2 e^{-t} - \frac{4}{5} C_2 e^{\frac{-4}{5}t} + A_2 + B_2 e^{-t} + C_2 e^{\frac{-4}{5}t} = 1$$

$$-B_1 e^{-t} - \frac{4}{5} C_1 e^{\frac{-4}{5}t} + 2A_1 + 2B_1 e^{-t} + 2C_1 e^{\frac{-4}{5}t} + B_3 e^{-t} + \frac{4}{5} C_3 e^{\frac{-4}{5}t} + A_3 +$$

$$B_3 e^{-t} + C_3 e^{\frac{-4}{5}t} = 1$$

$$-B_2 e^{-t} - \frac{4}{5} C_2 e^{\frac{-4}{5}t} + A_2 + B_2 e^{-t} + C_2 e^{\frac{-4}{5}t} - B_3 e^{-t} - \frac{4}{5} C_3 e^{\frac{-4}{5}t} + 2A_3 +$$

$$2B_3 e^{-t} + 2C_3 e^{\frac{-4}{5}t} = 0$$

And by equalizing the coefficients of both sides, we get

$$A_1 = 3/4, A_2 = 1, A_3 = -1/2, B_1 = 0, B_3 = 0, C_2 = 4C_1, C_3 = (-2C_1)/3$$

therefore, the general solution is :

$$\therefore X(t) = \frac{3}{4} + C_1 e^{\frac{-4}{5}t}$$

$$Y(t) = 1 + B_2 e^{-t} + 4C_1 e^{\frac{-4}{5}t}$$

$$Z(t) = \frac{-1}{2} - \frac{2}{3} C_3 e^{\frac{-4}{5}t},$$

which has two arbitrary independent constants .

Example (2) : For solving the equations

$$X''(t) + Y''(t) - X'(t) + 2Y(t) = e^t$$

$$Y''(t) + Z''(t) + 2Y'(t) + Z'(t) = e^t$$

$$X''(t) + Z''(t) - X'(t) + Z'(t) = e^t$$

By taking Laplace transformation to both sides and without using any initial conditions, we find

$$(p^2 - p) X(p) + (p^2 + 2) Y(p) = \frac{h_1(p)}{p - 1}$$

$$(p^2 + 2p) Y(p) + (p^2 + p) Z(p) = \frac{h_2(p)}{p - 1}$$

$$(p^2 - p) X(p) + (p^2 + p) Z(p) = \frac{h_3(p)}{p - 1}, \quad \text{see [2]}$$

where $h_1(p), h_2(p)$ and $h_3(p)$ are polynomial of p , by using grammar rule, we get:

$$X(p) = \frac{K_1(p)}{p(p-1)^2}$$

where $K_1(p)$ is a polynomial of p and by taking $L^{-1}.T$ to both sides ,we get the formula of $x(t)$, and by the same way we can find the formula of $y(t)$ and $z(t)$ so:

$$\begin{aligned} X(t) &= A_1 + B_1 e^t + C_1 t e^t \\ Y(t) &= A_2 + B_2 e^t + C_2 e^{-2t} \\ Z(t) &= A_3 + B_3 e^t + C_3 e^{-t} \end{aligned}$$

which represent the general solution of required equations .

The given equations have six degree of Δ ,so the general solution must contain six constants, while the general solution formula of (L.S.O.D.E) has nine constants , therefore ,we shall eliminate three constants ,for this we get $x'(t), y'(t), z'(t), x''(t), y''(t)$ and $z''(t)$ formula from the general solution ,as follows:

$$\begin{aligned} X'(t) &= (B_1 + C_1) e^t + C_1 t e^t : Y'(t) = B_2 e^t - 2C_2 e^{-2t} \\ \text{and } Z'(t) &= B_3 e^t - C_3 e^{-t} \end{aligned}$$

$$\begin{aligned} X''(t) &= (B_1 + 2C_1) e^t + C_1 t e^t : Y''(t) = B_2 e^t + 4C_2 e^{-2t} \\ \text{and } Z''(t) &= B_3 e^t + C_3 e^{-t} \end{aligned}$$

By substituting $X(t), X'(t), X''(t), Y(t), Y'(t), Y''(t), Z(t), Z'(t)$ and $Z''(t)$ in the original equations ,we get

$$(B_1 + 2C_1) e^t + C_1 t e^t + B_2 e^t + 4C_2 e^{-2t} - (B_1 + C_1) e^t - C_1 t e^t + 2B_2 e^t - 4C_2 e^{-2t} = e^t$$

$$B_2 e^t + 4C_2 e^{-2t} + B_3 e^t + C_3 e^{-t} + 2B_2 e^t - 4C_2 e^{-2t} + B_3 e^t - C_3 e^{-t} = e^t$$

$$(B_1 + 2C_1) e^t + C_1 t e^t + B_3 e^t + C_3 e^{-t} - (B_1 + C_1) e^t - C_1 t e^t + B_3 e^t - C_3 e^{-t} = e^t$$

And by equalizing the coefficients of both sides, we get

$$C_1 = 1/2, B_2 = 1/6, B_3 = 1/4,$$

therefore, the general solution is :

$$\begin{aligned} X(t) &= A_1 + B_1 e^t + \left(\frac{1}{2} t e^t \right) \\ Y(t) &= A_2 + (e^t)/6 + C_2 e^{-2t} \end{aligned}$$

$$Z(t) = A_3 + (e^t)/4 + C_3 e^{-t}$$

This solution contains only six independent constants equal to the degree of Δ .

Example (3) : For solving the equations

$$X'(t) - 2X(t) - Y(t) = 0$$

$$Y'(t) - Y(t) = 6e^{-t}$$

$$Z'(t) - X(t) = 2\sin t ,$$

with the initial conditions $X(0) = Y(0) = Z(0) = 0$.

We take L.T to both sides and by substituting the initial conditions , we find

$$(p-2) X(p) - Y(p) = 0$$

$$(p-1) Y(p) = \frac{6}{p+1}$$

$$p Z(p) - X(p) = 0 ,$$

by using grammar rule, we get

$$X(p) = \frac{K_1(p)}{p(p-1)(p-2)(p+2)} , \text{see [1]}$$

where $K_1(p)$ is a polynomial of p .

We can find $X(t)$,after taking $L^{-1}.T$ of above equation and by the same way we can find $Y(t)$ and $Z(t)$ so

$$X(t) = A_1 + B_1 e^t + C_1 e^{2t} + D_1 e^{-t}$$

$$Y(t) = A_2 + B_2 e^t + C_2 e^{2t} + D_2 e^{-t}$$

$$Z(t) = A_3 + B_3 e^t + C_3 e^{2t} + D_3 e^{-t} + E_3 \cos t + G_3 \sin t$$

since the general solution of $X(t)$, $Y(t)$ and $Z(t)$ have fourteen constants so we need the same number of Linear equations ,we get some of them by the initial conditions $X(0)=Y(0)=Z(0)= 0$,and to find the other equations we shall find additional conditions by substituting the value of primary conditions in the original equations we find another initial conditions ,as follows :

$$X'(0) = 0 , Y'(0) = 6 , Z'(0) = 0 , X''(0) = 6 , Y''(0) = 0 , Z''(0) = 2 ,$$

$$X'''(0) = 12 , Y'''(0) = 6 , X^{(4)}(0) = 18 , Y^{(4)}(0) = 0 , Z^{(4)}(0) = 10 ,$$

$$Z^{(5)}(0) = 18 .$$

by substituting these initial conditions in the derivatives of the general solution of $X(t)$, we get

$$X(0) = A_1 + B_1 + C_1 + D_1 = 0$$

$$X'(0) = B_1 + 2C_1 - D_1 = 0$$

$$X''(0) = B_1 + 4C_1 + D_1 = 6$$

$$X'''(0) = B_1 + 8C_1 - D_1 = 12$$

By solving these equations, we find

$$A_1=0, B_1=-3, C_1=2, D_1=1$$

Also, by the same way we can find the other constants, we get

$$A_2=0, A_3=5, B_2=3, B_3=-3, C_2=0, C_3=1, D_2=-3, D_3=-1, E_3=-2, G_3=0$$

And by substituting these values of constants in $X(t), Y(t)$ and $Z(t)$, so:

$$X(t) = -3e^t + 2e^{2t} + e^{-t}$$

$$Y(t) = 3e^t - 3e^{-t}$$

$$Z(t) = 5 - 3e^t + 2e^{2t} - e^{-t} - 2\cos t$$

which is represent the particular solution of the original equations.

Example (4) : To solve the set of the L.S.O.D.E

$$X''(t) + 2X'(t) + X(t) + 2Y'(t) + 3Z'(t) = 1$$

$$X'(t) + Z(t) = 0$$

$$X(t) - Y'(t) - Z'(t) = 0, \text{ with the initial conditions}$$

$$X(0) = Z(0) = 1, Y(0) = 0.$$

We take L.T to both sides and by substituting the initial conditions, we find:

$$(p^2 + 2p + 1)X(p) + 2pY(p) + 3pZ(p) = \frac{p^2 + 4p + 1}{p}$$

$$pX(p) + Z(p) = 1$$

$$X(p) - pY(p) - pZ(p) = -1$$

By using grammar rule, we get

$$X(p) = \frac{K_1(p)}{p(p + \frac{3}{2})}, \text{ see [1]}$$

where $K_1(p)$ is a polynomial of p .

After that we take $L^{-1}.T$ to both sides to above equation, we get

$$X(t) = A_1 + B_1 e^{\frac{-3}{2}t},$$

by the same way we can find $Y(t)$ and $Z(t)$, so

$$Y(t) = A_2 + B_2 t + C_2 e^{\frac{-3}{2}t}$$

$$Z(t) = A_3 + B_3 e^{\frac{-3}{2}t},$$

since the general solution of $X(t)$, $Y(t)$ and $Z(t)$ have seven of constants so we need the same number of linear equations, we get some of them by the initial conditions $X(0)=Z(0)=1, Y(0)=0$, and to find the other equations we shall find additional conditions by substituting the values of primary conditions in the original equations we find another initial conditions, so:

$$X'(0) = -1, Y'(0) = 5/2, Z'(0) = -3/2, X''(0) = (23/2), Y''(0) = (-13/4), Z''(0) = 9/4,$$

Also, we derive $X(t)$ to the same numbers of initial conditions that we found and by substituting the initial conditions in the derivatives, we get:

$$X(0) = A_1 + B_1 = 1, \quad X'(0) = (-3/2) B_1 = -1$$

by solving these two equations, we find $A_1 = 1/3, B_1 = 2/3$.

Also by the same way, we can find the other constants, so

$$A_2 = (13/9), B_2 = 1/3, C_2 = (-13/9), A_3 = 0, B_3 = 1$$

And by substituting these values of constants in $X(t), Y(t)$ and $Z(t)$, we get :

$$X(t) = \frac{1}{3} + \frac{2}{3} e^{\frac{-3}{2}t}$$

$$Y(t) = \frac{13}{9} + \frac{1}{3}t - \frac{13}{9} e^{\frac{-3}{2}t}$$

$$Z(t) = e^{\frac{-3}{2}t}$$

So $X(t)$, $Y(t)$ and $Z(t)$ represent the particular solution of the original (L.S.O.D.E).

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