التقريب الخطي لحساب
التكاملات المعتلة مع تقدير الخطأ

Linear approximation to evaluation singular integrals and error estimation

المدرس المساعد
علية شاني حسن
جامعة القادسية / كلية العلوم والرياضيات

Alia Shani Hassan
University Of Al-Qadisiya
College Of Computer Science & Mathematics

التقريب الخطي لحساب التكاملات المعتلة مع تقدير الخطأ

Linear approximation to evaluation singular integrals and error estimation

علية شاني حسن
جامعة القادسية / كلية العلوم والرياضيات

Alia Shani Hassan
University Of Al-Qadisiya
College Of Computer Science & Mathematics

Abstract:

In this paper we get method to evaluate singular integrals by using linear approximation of integrand for treatment singularity at one of or both infinite intervals integral. Numerical example is given to show the practical application of the method.

I- Introduction

Let R[I] be the set of real functions defined on the closed interval I=[a,b] on the real axis and let T[f] be a linear function over R[f] ,i.e. a linear function defined for every $f(x) \in R[I]$, we assume the continuity to T(f) in the sense that

T[fn]→ 0 as $f_n \in R[I]$ (1)

Approaches uniformly to zero with its all derivatives.
The function $f(\alpha)$ *at* $\alpha \in I$, the derivative $f'(\alpha)$ *at* $\alpha \in I$, and the definite Integral over I are continuous linear functions over R[I] , for example .

A continuous linear function over R [I] can be expressed in the form of a contour integral

$$T[f] = \int \psi(x) f(x) dx, \qquad (2)$$

where

$$\psi(x) = T\left[\frac{1}{x-\alpha}\right], \qquad (3)$$

If $f \in R[I]$ there exists an open interval J which includes I.
The relation (2) can be easily obtained if we substitute the Cauchy's representation of function

$$f(x) = \int \frac{1}{x-\alpha} f(x) dx \qquad (4)$$

In to T[*f*] and exchange the order of the operation T and the integration.
This exchange is showed to permitted as follows.
The integral in the right hand side of (4) can be approximated uniformly for
$\alpha \in I$ by a Riemann sum

$$f_n(\alpha) = \sum_{k=1}^{n} \frac{1}{\lambda_k - x} f(\lambda_k) \Delta\lambda_k, \qquad (5)$$

where λk is a point in I and $\Delta\lambda_k = \lambda_k - \lambda_{k-1}$.

Since λ_k and $\Delta\lambda_k$ do not depend on x and T[f] is linear with respect to f ,we have

$$T_x[f_n] = \sum_{k=1}^{n} T_x\left[\frac{1}{\lambda_k - x}\right] f(\lambda_k)\Delta\lambda_k, \quad , \qquad (6)$$

The subscript x is attached to T in order to show explicitly that the operation of T is that with respect to x .When n→∞ ,f_n approach to f and we have

$$\lim_{n\to\infty} T\left[f - f_n\right] = 0 \quad , \qquad (7)$$

from the continuity of T, and hence

$$\lim_{n\to\infty} T[f_n] = \int T\left[\frac{1}{x-\alpha}\right] f(x)dx \qquad (8)$$

There are many approximation formulas $T_n[f]$ for numerical calculation which can be regarded as continuous linear function on R[I] . For such formulas we also have

$$T_n[f] = \int \Psi_n(x) f(x)dx, \qquad (9)$$

$$where \qquad \Psi_n(x) = T_n\left[\frac{1}{x-\alpha}\right]. \qquad (10)$$

Then the error En[f] which is produced when T[f] is approximated by $T_n[f]$ is given

$$E_n[f] = T[f] - T_n[f] = \int \phi_n(x) f(x)dx \,, \qquad (11)$$

$$where \qquad \phi_n(x) = \psi(x) - \psi_n(x) \;, \qquad (12)$$

The function $\phi_n(x)$ is determined only by T and T_n , and does not depend on f.

We call $\phi_n(x)$ the characteristic function of the error .

2-linear approximation of real function

The Contrul integral representation of T[f]=f(x) is nothing but the Cauchy's representation (4), and hence we have

$$\Psi(x) = \frac{1}{x-\alpha} \tag{13}$$

In the present problem
Let $T_n[f]$ be a linear approximation of f(x). Then the error is given by

$$E_n[f] = \int \phi_{n,\alpha}(x) f(x) dx, \tag{14}$$

$$\text{where} \qquad \phi_{n,\alpha}(x) = \frac{1}{x-\alpha} - T_n[\frac{1}{x-\alpha}] \quad . \tag{15}$$

Since the error depends on the point x at which the evaluation of Tn[f] is carried out the characteristic function $\phi_{n,\alpha}(x)$ also depends on α .For the maximum error

$$E_n^{\max}[f] = \max_{a\le\alpha\le b} |E_n[f]| \quad , \tag{16}$$

we have an estimate

$$E_n^{\max}[f] \le \int \phi_n^{\max}(x) |f(x)| |dx|, \tag{17}$$

Where

$$\phi_n^{\max}(x) = \max_{a\le\alpha\le b} |\phi_{n,\alpha}(x)|, \tag{18}$$

And it does not depend on x.

3-Error Estimation in Taylor's series

The truncation error of the following Taylor's series for a function f(x) around $x=x_0$

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + ... + \frac{1}{(n-1)!}(x - x_0)^{n-1} f^{(n-1)}(x_0) + E_n(f)$$

(19)

can be expressed in the form of a contour integral

$$E_n[f] = \int \phi_{n,\alpha}(x) f(x) dx \qquad (20)$$

$$where \qquad \phi_{n,\alpha}(x) = \frac{(\alpha - \alpha_0)^n}{(x-\alpha)(x-\alpha_0)^n}, \qquad (21)$$

we put $\alpha_0 = 0$ and assume I=[0,1],i.e. we assume that the truncated Taylor's series is used in the range [0,1] in the actual computation .then

$$|\alpha_n(x)| = \frac{|\alpha|^n}{|x-\alpha||\alpha|^n} \qquad (22)$$

4-Polynomial Interpolation

The error of polynomial interpolation $f_n(x)$ with the sampling points $x_1, x_2, \ldots, x_n$ for a function $f(x)$ is given by

$$E_n[f] = f(\alpha) - f_n(\alpha) = \int \phi_{n,\alpha}(x) f(x) dx, \qquad (23)$$

in which the characteristic. Function is defined by

$$\phi_{n,\alpha}(x) = \int_{\alpha}^{\alpha+\varepsilon} \frac{\psi(x)}{(x-\alpha)} dx$$

after replacing ψ by its p-th order Taylor's expansion around $x=\alpha$,we obtain

$$\psi(x) = \omega_p(x) + \frac{(x-\alpha)^{p+1}}{(p+1)!}\psi^{(p+1)}(\lambda(x)) \qquad , p \geq 0$$

(24)

$$where \qquad \omega_p(x) = \sum_{k=0}^{p} \psi^k(\alpha)(x-\alpha)^k / k!$$

Then $$\phi_{n,\alpha}(x) = \varepsilon^{1-M}\sum_{k=0}^{p}\frac{\varepsilon^k \phi^k(a)}{k!(k+1-M)} + \frac{1}{(p+1)!}\int_{\alpha}^{\alpha+\varepsilon}(x-\alpha)^{p+1-M}\phi^{P+1}(\lambda(x))dx.$$
(25)

5-Singular Integrals

We will consider the case of integrals of bounded function over unbounded intervals.

Let us deal with the $\lim_{x\to a^+} f(x) = \infty$ or undefined, similar consideration hold when case in which f is infinite as $x\to b^-$, while the case of a point of singularity c internal to the interval [a,b] can be recast to one of the previous two cases owing to

$$I(f) = \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (26)$$

assume that the integrand function is of the form

$$f(x) = \frac{\phi(x)}{(x-a)^{\lambda}}, \qquad 0 \le \lambda \le 1$$

Where ϕ is a function whose absolute value is bounded by M . Then

$$|I(f)| \le M \lim_{t\to a^+}\int_t^b \frac{1}{(x-a)^{\lambda}}dx = M\frac{(b-a)^{1-\lambda}}{1-\lambda}$$

Example: $I = \int_0^1 \frac{1}{(1-x)}$ note that ,the integrand is singular at x=1 ,to treatment

that, assume that r(x)=a+(b-a)x

then $$x(r) = \frac{r-a}{b-a},$$

and hence $$dx = \frac{1}{b-a} dr$$

therefore $$I = \int_{2}^{3} \frac{1}{1-\frac{r-2}{3-2}} dr$$

We obtain an approximate value of integral equal to -0.28768

References

[1]-P.J. Davis: "Interpolation and approximation ", Blaisdell, 1963.

[2]-R. Sacco: “Numerical Mathematics", 2000.

[3]- H. Takahasi and M. Mori: “Error Estimation in the Numerical Integration of Analytic Functions” ,report of the computer Centre, Univ .of Tokyo (1970).

[4]- Jintae K.," Interpolation and Polynomial approximation”,(2003).

التقريب الخطي لحساب التكاملات المعتلة مع تقدير الخطأ

الملخص:

في هذا البحث تمكنا من الوصول إلى طريقة لحساب التكاملات المعتلة باستخدام التقريب الخطي للدالة المكاملة ومعالجة الاعتلال إذا كان في إحدى أو كلتا نهايتي فترة التكامل والطريقة موضحة في البحث مع إعطاء مثال عددي للتوضيح.

$$I(f) = \int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$

ونفرض أن الدالة المكاملة تأخذ الشكل الأتي: $f(x) = \frac{\phi(x)}{(x-a)^{\lambda}}, \quad 0 \le \lambda \le 1$

فأن

$$|I(f)| \le M \lim_{t \to a^+} \int_{t}^{b} \frac{1}{(x-a)^{\lambda}} dx = M \frac{(b-a)^{1-\lambda}}{1-\lambda}$$

تطرق البحث إلى متسلسلة تايلور وتقدير الخطأ في هذه المتسلسلة وكيفية استخدامها في اندراج متعددات الحدود.