On the shape of the numerical range of similar operators and the unitarily invariance

of the joint numerical range

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Abstract:-

P.R.halmos [5] has proved that similar operator on a Hilbert space H have the same spectrum, moreover they have the same point spectrum, approximate point spectrum and the same compression spectrum .

In this paper we show that the situation is absolutely different for the numerical range by giving an example.

Theorem.1/ gives a necessary condition on the similar operator S and T get that $\overline{W(S)} = \overline{W(T)}$, where $\overline{(\ )}$ denotes closure, we also consider the case when H is finite dimensional.

In theorm.2/ we prove that if T and S are similar operators on complex Hilbert space H such that W(T) is a closed polygon then W(S) contains the interior of W(T) , while when W(S) and W(T) are both closed polygon then W(S)=W(T).

Moreover we prove theorm.3/which states that the numerical range of any operator Ton H unitarily invariant (i.e. W (U*TU) =W (T)), and we conclude that the numerical radius is also unitarily invariant.

Finally, we extend thm.3/ to the case of n- tuples of operators on H.

From theorm.4/ we conclude that the joint numerical radius is also unitarily invariant.

Notation:-

In what follows, H denote a complex Hilbert space and T is any operator on H (bounded liner transformation T: H-----→H, or we say T□B(H)). The numerical range W (T)

Of T□B(H) is defined as:-

W (T) = {<Tx,x>**:**x□H,‖x‖=1},

Which is already compact when H is finite dimensional [4], we also know that the convex hull conh (6(T)) of 6(T) of T□B(H) is identical with $\overline{W(T)}$ when T is normal (or subnormal)[5]

The notation π□(T) denote the point spectrum of T , which consist of the set of all eigen values of T , while π(T) denotes the approximate point spectrum and define as :-π(T)={λ□□:□ a seq. {x_n} of unit vectors in H such that:}

‖ (T-λI)x_n‖--→0 as n--→∞.

The compression spectrum □ (T) of T□B (H) is defined as □ (T) = {λ□□: range (T-λI) ≠H}

Now, we will give our main results obtained in this paper in three sections.

First section/ gives an example of 2 similar operators S and T on H such that W(S) ≠W (T), we also give a necessary condition an any 2 similar operators S and T on H to get that $\overline{W(T)}$ =$\overline{W(S)}$, we also consider the case when W (T) and W(S) are both closed polygons and we prove that W(T)=W(S) in this case. We also consider the case when one of W (T) or W(S) is a closed polygon and prove that the other contain P.

In section 2/ we prove that the numerical range W (T) of T□B (H) is unitarily invariant, and we conclude that the numerical radius is also unitarily invariant.

Finally, in section 3/ we generalize the results of the previous section to the case of n-tuples of operators on H, where the joint numerical range W(A) and the joint numerical radius w(A) of $A=(A_1,A_2,\dots A_n)$ are define as:-

$W(A) = \{(<A_1^x, x>, <A_2^x, x>\dots<A_n^x,x>):x\square H,\|x\|=1\}$

$w(A) = \sup. \{|z| :z\square W(A)\}$

Main results

Section.1

Theorem.A.[5, page 39]

For any two similar operators S and T in B(H),

$\sigma(S)=\sigma(T)$. moreover;

1-$\pi_\circ(S)=\pi_\circ(T)$

2-$\boldsymbol{\pi}(S)=\boldsymbol{\pi}(T)$

3-$\Gamma(S)=\Gamma(T)$

Now, we need to show that the above fact for similar operators will not valid for the numerical range (I.e. W(S)) and W(T) may be different), by giving the following example.

Example:

Let be a 2-dimentional complex Hilbert space , and let

$T=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, S=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$

Clearly T and S are similar , since $\exists A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ such that

$S=A^{-1}\,T\,A$

Now, we have

$W(T)=\{ <Tx,x>: x\in H,\|x\| =1\}$

$=\{< \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\binom{\propto 1}{\propto 2}, \binom{\propto 1}{\propto 2} > : x=\binom{\propto 1}{\propto 2} \in H, \|x\| = (\sum_{i=1}^{2}|\propto i|^2)^{1/2} =1\}$

$=\{< \binom{\propto 1}{0}, \binom{\propto 1}{\propto 2} >: x = \binom{\propto 1}{\propto 2} \in H, \ |\propto 1|^2 + |\propto 2|^2 =1\}$

$=\{\propto 1.\overline{\propto 1} + 0.\overline{\propto 2} : \ |\propto 1|^2 + |\propto 2|^2 =1, \ x=\binom{\propto 1}{\propto 2} \in H\}$

Because $< x, y > = \sum_i < x, ei >\overline{< y, ei >}$,when [ei] is a complete orth .normal .set for H and

$.< x, ei > =\propto i$

$=\{|\propto 1|^2 : |\propto 1|^2 + |\propto 2|^2 =1, x=\binom{\propto 1}{\propto 2} \in H$

W(T)=[o, 1] , and

W(S)=$\{< Sx, x >: x \in H, \|x\| = 1\}$

$=\{< \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\binom{\propto 1}{\propto 2}, \binom{\propto 1}{\propto 2} > : \ |\propto 1|^2 + |\propto 2|^2 =1, \ x=\binom{\propto 1}{\propto 2} \in H\}$

$=\{ \ < \binom{\propto 1+\propto 2}{0}, \binom{\propto 1}{\propto 2} >:: \ |\propto 1|2 + |\propto 2|2 = 1, \ x = \binom{\propto 1}{\propto 2} \in H\}$

$=\{ (\propto 1+\propto 2).\overline{\propto 1} + 0.\overline{\propto 2} : \ |\propto 1|^2 + |\propto 2|^2 =1, \ x=\binom{\propto 1}{\propto 2} \in H\}$

$=|\propto 1|^2 +\propto 2.\overline{\propto 1} : \ |\propto 1|^2 + |\propto 2|^2 =1, \ x=\binom{\propto 1}{\propto 2} \in H\}$

Now, if we take z $\in$ W(S) and consider Im(z) the imaginary part of z , we have:

Im(z)=Im($|\propto|^2$ +Im($\propto 2.\overline{\propto 1}$) , but $|\propto 1|^2$ is real

=Im($\propto 2 \overline{\propto 1}$)

If we take $\propto 1 = 1/2e^{i\theta}$ $((i.e.)$with $|\propto 1| = 1/2)$, and

.$\propto 2 = |\propto 2|.\, e^{i\emptyset}$,with$\emptyset \neq \theta$,we get

.$|\propto 2|=\sqrt{1-(\frac{1}{4})}=\sqrt{3}/2$

I.e.$\propto 1 = \frac{1}{2}e^{i\theta}$

$\propto 2 = \sqrt{3}/2e^{i\emptyset}$

Now,

Im (z)=Im($\propto 2\ \overline{\propto 1}$)

=Im ($\sqrt{3}/2.e^{i\emptyset}.1/2.e^{-i\theta}$)

=Im($\sqrt{3}/2\ .\ e^{i(\emptyset-\theta)}$

But, $\emptyset - \theta \neq 0$

Thus, Im$e^{i(\emptyset-\theta)}$≠0

.$\Rightarrow Im \quad (z) \neq 0$

i.e. z is not real

⇒zϵ [0,1]= WT

⇒W(T)≠ W(S)

In fact, one can compute W(S) as in [3] and can find that W(S), is the ellipse with foci 0 and 1 and eccentricity sinθ, where θ *is the angle* between the two Eigen vectors of S corresponding to the Eigen values 0 and 1

Theorem 1.

Let T and S be similar operators on a Hilbert space H, if T and S are subnormal, then:

.$\overline{W(S)}=\overline{W(T)}$

Proof:

Since T and S are subnormal, we get that [5], [1] Conh(σ(T))= $\overline{W(T)}$ and

Conh (σ(S))=$\overline{W(S)}$,where Conh denotes the convex hull

But, T and S are similar , we get

σ(T)=σ(S)

Therefor

.$\overline{WT}$ =Conh (6(T)) = conh (6(S)) =$\overline{W(S)}$.

<u>Corollary.</u>

Let T and S be similar operators on a finite dimensional Hilbert space H.If T and S are subnormal then

W(T)=W(S)

Proof

By applying theorem 1, we get that:

.$\overline{W(T)}$= $\overline{W(S)}$ ………………………………………1.1

Since H is finite dimensional, we conclude that the unit sphere S(H)={ x∈ H: $\|x\| = 1\}$ of H is compact.

Now, W(T) is a continuous image of S(H), we will get that W(T) is compact ∀ T ∈ B(H).

But we know that ,every compact subset of a Hausdorff space is closed [6, page 153, theorem 11.5]

i.e.($\overline{W(T)}$=W(T) and $\overline{W(S)}$=W(S)

therefore W(T)=W(S).

Now, we will take a geometric characterization for the numerical range of similar operators,specially when the numerical range of one of these operators (or both of them) is closed polygon.

<u>Theorem .B</u> ,[7 , corollary 2.1]

For a complex Hilbert space H , and T϶ B (H), If$\overline{W(T)}$ is a closed polygon then Conh (σ(T))≼$\overline{W(T)}$

<u>Lemma:</u>

Let T and S be similar operators on, we have:

1-If$\overline{WT}$ = closed polygon P , then $\overline{W(S)}$contains P (and thus W(S) contains P°)

2-If $\overline{WT}$and $\overline{WS}$ are closed polygon then $\overline{W(T)}$= $\overline{W(S)}$

Proof:

Suppose that $\overline{W(T)}$= closed polygon P.

now , by applying theorem . B we get that :

C on h (σ(T))= $\overline{W(T)}$…………………………………(1.2)

But, S and T are similar , from theorem . A we get that :

σ(T) =σ(S)……………(1.3)From (1.2) and (1.3) we get that:

Conh(6 (S))= $\overline{W(T)}$=P………………………………(1.4)

But, we know that [5] ,the closure of the numerical range of any operator contains the spectrum of that operator , i.e.

6(S) ≤ $\overline{W(S)}$) , ∀ S ϵ B(H)

But $\overline{W(S)}$ is always convex [5].

Therefore

P=ConH (6 (S)) ≤ $\overline{W(S)}$ ………………….……….(1.5)

From (1.4) and(1.5) we get that

$$\overline{W(T)} \leq \overline{W(S)}$$

i.e. P ≤ $\overline{W(S)}$,

P°≤ $\overline{(W(S))}$ ° , where ( )° denotes interior

But, a theorem of Bouldin [2] , states that:

[$\overline{W(S)}$]°≤ W(S) , ɏ S ϵ B (H).

Thus, P° ≤W(S).

Suppose that$\overline{W(T)}$ =P1 ………………………(1.6)

$\overline{W(S)}$ = P2 ……………………………..………(1.7)

Appling first part of Lemma on equation (1.6) we get

P1 ≤ $\overline{W(S)}$ =P2 ………………………………….(*)

Again, we apply the first part of the Lemma on equation (1.7)

We will get

P2 ≤ $\overline{W(T)}$ = P2 ………………………………(**)

From (*) and (**) we get that : P1=P2

i.e.$\overline{W(T)}$ =$\overline{W(S)}$.

<u>Theorem .2</u>

Let T and S be any two similar operator on a complex Hilbert space H ,we have :

If W(T) is a closed polygon then W(S) contains :P° 1-

If W(T) and W(S) are closed polygon then W(T)=W(S) 2-

<u>Proof</u>

Follow directly from the Lemma , under assumption that W(T) and W(S) are closed .

Section .2:

Let S and T be operators on Hilbert space H , a basic problem is the determination of W(ST). In [4],

A super set for W(T) is given.

<u>Remark.1</u> [5]

The inequality w(ST) ≤ W(S).W(T) is not true when we take: S=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,T=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have w(S) =w(T) = ½ ,but w(ST) = 1

Remark :2 [5]

If S and T are normal operator on H,then:

w(ST)≤w(S).w(T)

Proof

Since S and T are normal , we have:

w(S)=$\|S\|$, w(T)=$\|T\|$

Now,

W(ST) ≤ $\|ST\| \leq \|S\| \cdot \|T\|$ =w(S) .w(T)

Section.2

In this section, we prove that the numerical range is unitarily invariant, and similarly for the numerical radius.

Theorem.3

Let T be any operator on a Hilbert space H, if U is a unitary operator on H, then

W(U^* T U)= W(T)

Proof:

W(U^*TU)= $\{ < (U^*TU)x ,x >: x \in H , \|x\| = 1\}$

$=\{ < T\, Ux, Ux \ >: x \in H, \|x\| = 1\}$

But $\|Ux\| = \|x\|, \forall\, x \in H$

$\Rightarrow \|Ux\| = \|x\| = 1\ \forall\, x\ \in S(H) = sphere\ of\ H$

$\Rightarrow \|W\|(U^{\times}TU) = \{ < T\, Ux, Ux >:$ Ux ∈ H, $\|x\|$=1}

W($U^{\times}TU) \ \leq W(T)$ …………………….(2.1)

By Appling (2.1) on $U^{\times}$ since it is also unitary ,we get

W(U.$U^{\times}$TU .$U^{\times}$) ≤ W($U^{\times}$TU)

⇒ W(T) ≤ W ($U^{\times}$TU) ………………….(2.2)

From (2.1) and (2.2) we conclude that:

W(U×TU) =W(T)

Corollary:

Let T be operator on H .If U is a unitary operator on H, then:

w(U×TU)=w(T)

Proof

It follows directly from theorem. 3

Section.3:

In this section we generalize the results of section .2 to the case of n-tubes of operators on H,

Theorem 4

Let A=(A1,A2,……..,An) be any n-tuple of operators on H and let U be a unitary operator on h, then

W(U×AU)=W(A)

Proof

By definition, we have:

W(U×AU)={(U×$A1U)x,x>,<(U^{\times}A2U)x,x>,\ldots.,<(U^{\times}AU)x,x>):x\in H1\|x\|=1$}

={(< $A1Ux,Ux>,A2Ux,Ux>,\ldots\ldots.,<A_n$ Ux, Ux>): x ∈ H,‖x‖=1}

But ,‖Ux‖ =‖x‖=1 ,∀ x ∈ S(H)

,

⇒ W(U×AU)={(A_1Ux, Ux>,< A_2Ux, Ux > , … … . . < A_n Ux, Ux>): $\|Ux\|=1, Ux\in H$}

⇒W(U×AU)≤ W(A)……………………….…….(3.1)

Applying 3.1 for U× since it is also unitary , we get:

W(U.U×AU.U×) ≤ W(U×AU)

$\Rightarrow W(A) \leq W(U^{\times}AU)$……………………………….(3.2)

From(3.1) and (3.2) ,we get:

$W(U^{\times}AU)=W(A)$:

<u>Corollary:</u>

Let A= (A1 , A2 ,… , A_n) be any n-tuple of operators on H ,and U be any unitary operator on H,then :

W(U*AU)= W(A)

<u>Proof</u>

It follows directly from theorem 4

الملخص:

تم في هذا البحث الحصول على الشرط الضروري على مؤثرين متشابهين T,S على فضاء هلبرت H لكي يكون لهما نفس الإنغلاق للمدى العددي للمؤثرين أعلاه $\overline{W(S)} = \overline{W(T)}$, كذلك تحت دراسة المدى العددي للمؤثرين المتشابهين على فضاء هلبرت و استنتجنا بأن :-

أولاً. إذا كان $\overline{W(T)}$ مضلع مغلقفإن $\overline{W(S)} \geqslant [W(T)]^{\circ}$=(داخل المدى العددي للمؤثر T)

ثانياً. إذا كان كل من W(S) وW(T) مضلع مغلق فإن W(S)=W(T) بالإضافة لذلك استنتجنا بأن

ثالثاً. W(U*TU)=W(T) حيث U هو مؤثر unitary operator

رابعاً. تم تعميم المبرهنة السابقة في حالة n-tuple

خامساً. مع مبرهنات اخرى.

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